On The Combinatorial Properties of Nilpotent and Idempotent Conjugacy Classes of the Injective Order-Preserving Transformation Semigroup

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Abstract

We presented the elements of the injective order-preserving transformation semigroup using path structure (Circuits and Proper paths) introduced by Stephen Lipscomb. We grouped the elements according to their Conjugacy class, identified the nilpotents and the idempotents and studying the nilpotent and idempotent conjugacy class chain decompositions. Some observations were made and their order enumerated. Formulae were found for their order and combinatorial properties

Keywords: Combinatorial properties, Nilpotent conjugacy classes, Idempotent conjugacy classes, Injective Order-preserving Transformation semigroup

1. INTRODUCTION

Semigroup of injective order-preserving transformations, a subsemigroup of the symmetric inverse semigroup (I_n) was introduced by Garba(1994) and defined as thus:

Let α be a transformation in I_n, it is said to be of order-preserving if $(\forall x, y \in dom\ \alpha) x \leq y \Longrightarrow x\alpha \leq y\alpha$. The order of the injective order-preserving transformations(IO_n) corresponds to the sequence 2, 6, 20, 70, 252, 924, 3432, … when n = 1, 2, …. A formula for its order is given as

$IO_n = \binom{2n}{n}$. It is well known that an element $x \in I_n$ is nilpotent $(x^n = 0)$ for some $n > 0$. A property of nilpotent element among others is $x\alpha \neq x \; \forall \; x \in dom\alpha$.

In this paper, we are adopted the path notations invented by Lipscomb(1996) for I_n which he defined as follows
Let $\mathbb{N} = x_1, x_2, \dots, x_m$ and $\alpha \in I_n$ have domain $d\alpha = x_1, x_2, \dots, x_m$ and if $x_1\alpha = x_2, x_2\alpha = x_3, x_{m-1}\alpha = x_m, x_m = y$. Then α is a path. Having a circuit or a proper path depends on the value of y. If $y = x_1$ then $\alpha = (x_1, x_2, \dots, x_m)$ is a circuit of length m. If $y \neq x_1$ then $\alpha = (x_1, x_2, \dots, x_m, y]$ is a proper path of length (m + 1)
Various text or papers have slightly different path notations as can be seen in Munn(1957) where he used the notations "links" and "cycles" for proper paths and circuits respectively. He would write $(12)(345] \in I_5$ as $(12)[345]$ where (12) is the cycle while [345] is the link. Gomes and Howie(1987) denoted a primitive nilpotent as "‖1, 2, …, m‖" while Sullivan(1987) in his study of semigroups generated by nilpotent transformations denoted a proper path of length (m + 1) as m-chains [1, 2, …, m + 1] and a circuit of length m as m-cycles (1, 2, …, m)

Let G be a group. An element $x \in G$ is said to be conjugate to an element $y \in G$ if there exists $g \in G$ such that $y = g^{-1}xg$. Since conjugation is important to group theory, it was only quite natural to have extended it to some certain classes of semigroups.

Theorem 1.1
Let $x, y \in I_n$. Then the following holds:

a. x is conjugate to y if and only if they have the same path structure
b. x is nilpotent if and only if its path structure are joins of only proper paths.
c. x is idempotent if and only if all the paths in its path structure is of length one

Proof:

a. Let $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & - \end{pmatrix} \in I_6 = (123)(456]$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & - & 5 & 6 & 4 \end{pmatrix} \in I_6 = (123](456)$. a and b has a circuit of length 3 and a proper 3-path and hence falls in the same conjugacy class. Also we can find a permutation $g \; in \; permutation \; group$ such that $y = g^{-1}xg$ or $gy = xg$ by matching the paths in a vertical order

$$a = (123)(456]$$
$$b = (456)(123]$$

from where can deduce $g = (14)(25)(36)$

b. Let φ be a proper path and σ a circuit. A circuit is an extention of a permutation and permutation group has no nilpotents, therefore $x = \varphi\sigma \implies x^n \neq 0$. Hence x is nilpotent iff its path structure are only proper paths
c. Assuming $\varphi \; or \; \sigma$ has length 2, $x\alpha \neq x \; \forall x \in dom\alpha$. This implies that $x\alpha^2 \neq x\alpha$.

The definition of Conjugacy in arbitrary semigroups seems not to be unique as can be seen in Kudryavtseva and Mazorchuk(2009) where they compared three approaches of Conjugacy on semigroups. Dauns (1989) gave a definition for monoid and Lallement(1979) for free semigroups.

2. MONOGENIC SUBSEMIGROUP

Let S be a semigroup and let $a \in S$, then the monogenic subsemigroup $\langle a \rangle$ consists of all elements of S that can be expressed as positive integral powers of a. Here we say that a is a generating set of S. In this paper, we only consider the finite monogenic subsemigroup. If repetitions occur in the positive powers of a, then we have that $a^m = a^m + a^r = a^t$ where m is called the index, r the period and $m < t$. We have that the powers $a, a^2, \ldots, a^m, \ldots, a^{t-1}$ are distint and therefore $order\ of\ a = m + r - 1$. Below is a monogenic subsemigroup generated by $a = (12](34567)(8) \in I_8$

$$a = (12](34567)(8);\ a^2 = (1](2](35746)(8);\ a^3 = (1](2](36475)(8);$$
$$a^4 = (1](2](37654)(8);\ a^5 = (1](2](3)(4)(5)(6)(7)(8);\ a^6 = (1](2](34567)(8);$$
$$a^7 = (1](2](35746)(8)$$

From the illustration above, we can see that $a^2 = a^2 + a^5 = a^7$. It is worthy to note that the term 'Monogenic' was introduced by Howie(1995) as against the term 'cyclic' used by Clifford and Preston(1961) and Lipscomb(1996). Howie claimed that it did not merit the name 'cyclic' because it is not always cyclic as in the case of a singleton generator.

Theorem 2.1

Let $x \in I_n$. Then the following holds

a. The index of x is the maximum of lengths of all the proper paths in it. The index is one if no proper path exist in x.
b. The period of x is the lowest common multiple of all the lengths of the circuits in x. The period is one if no circuit exist in x.

Proof: See Lipscomb(1996) pp 13.

3. METHODOLOGY

The conjugacy classes of $\alpha \in IO_n$ were arranged according to the fix of α denoted as $f(\alpha)$ and defined by $f(\alpha) = |F(\alpha)| = |\{x \in X_n : x\alpha = x\}|$ for any number of IO_n. The nilpotent conjugacy classes are marked asterick($*$), while the idempotent conjugacy classes are marked at(@). The Index and period of each conjugacy class was also found. Some illustrations are given below

Conjugacy classes of IO_1

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1]*@	1	1
1	(1)@	1	1

Table 3.1

Conjugacy classes of IO_2

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2]*@	1	1
	(12]*	2	1
1	(1)(2]@	1	1
2	(1)(2)@	1	1

Table 3.2

Conjugacy classes of IO_3

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2](3]*@	1	1

	(12](3]*	2	1
	(123]*	3	1
1	(1)(2](3]@	1	1
	(1)(23]	2	1
2	(1)(2)(3]@	1	1
3	(1)(2)(3)@	1	1

Table 3.3

Conjugacy classes of IO_4

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2](3](4]*@	1	1
	(12](3](4]*	2	1
	(12](34]*	2	1
	(123](4]*	3	1
	(1234]*	4	1
1	(1)(2](3](4]@	1	1
	(1)(23](4]	2	1
	(1)(234]	3	1
2	(1)(2)(3](4]@	1	1
	(1)(2)(34]	2	1
3	(1)(2)(3)(4]@	1	1
4	(1)(2)(3)(4)@	1	1

Table 3.4

4. RESULT

We observed some combinatorial relations between numbers associated with the nilpotent and idempotent conjugacy classes of IO_n. We define the following numbers:

N_n = the cardinality of nilpotent conjugacy classes of IO_n

E_n = the cardinality of idempotent conjugacy classes of IO_n

M_n = the total number of chains in the nilpotent conjugacy class chain decomposition of IO_n

L_n = the total number of chains in the idempotent conjugacy class chain decomposition of IO_n

Table of combinatorial relations

n	N_n	E_n	M_n	L_n
1	1	2	1	2
2	2	3	3	4
3	3	4	5	6
4	5	5	7	8
5	7	6	9	10
6	11	7	11	12
7	15	8	13	14
8	22	9	15	16
9	30	10	17	18
10	42	11	19	20

Table 4.1

The following results are products of the table above

Lemma 5.1: Let $\alpha \in IO_n$, the nilpotent conjugacy classes of IO_n is given as $|N_n| = \frac{1}{192}n^4 - \frac{1}{16}n^3 + \frac{29}{48}n^2 - n + 2$ when n is even.

Proof: Using Mathematical Induction

We show that$|N_2| = 2$ is true $|N_2| = \frac{1}{192}2^4 - \frac{1}{16}2^3 + \frac{29}{48}2^2 = 2$

$|N_k|$ is true for some positive even integer: $|N_k| = \frac{1}{192}k^4 - \frac{1}{16}k^3 + \frac{29}{48}k^2 - k + 2$

We assume it holds for $|N_{k+2}| = \frac{1}{192}(k+2)^4 - \frac{1}{16}(k+2)^3 + \frac{29}{48}(k+2)^2 - k$

Then $|N_{k+2}| = \frac{1}{192}k^4 - \frac{1}{16}k^3 + \frac{29}{48}k^2 - k + 2 + \left(\frac{1}{24}k^3 - \frac{1}{4}k^2 + \frac{11}{6}k\right)$

$= \frac{1}{192}(k+2)^4 - \frac{1}{16}(k+2)^3 + \frac{29}{48}(k+2)^2 - k.$

Lemma 5.2: Let $\alpha \in IO_n$, the idempotent conjugacy classes of IO_n is given as $|E_n| = n + 1$.

Proof: For each idempotent rank of $n = 0, 1, \ldots, n \; in \; IO_n$, there exist at least one idempotent element. Also idempotent elements of a particular rank fall under a conjugacy class. Thus we have $n + 1$ idempotent conjugacy classes

Lemma 5.3: The total number of chains in the nilpotent conjugacy class chain decomposition of IO_n is $M_n = 2n - 1$.

Proof: Let α be a nilpotent transformation in IO_n with domain $X_n = 1, 2, \ldots, n$.

The number of images$(x\alpha) < X_n$ in any n of IO_n. Therefore number of total nilpotent conjugacy class chain decomposition of n will be $(n) empty \; maps + (n-1) \; maps$ of other combinations of chain decomposition of n. Hence $M_n = 2n - 1$

Lemma 5.4: The total number of chains in the idempotent conjugacy class chain decomposition of IO_n is $L_n = 2n$

Proof: From Theorem 1.1 we have that x is idempotent if and only if all the paths in its path structure is of length one. It implies that it is either a circuit of length one or a proper 1-path of n. Generally, there are two kinds of path in n ways. Therefore $L_n = 2n$

5. CONCLUSION

The sequence of $N_n = 1, 2, 3, 5, 7, 11, 15, 22, 30, 45, \ldots \; where \; n = 1, 2, \ldots$ is the partitions of a positive integer A000041 of the Online Encyclopedia of Integer Sequences(OEIS). Though we have only have general expression for when n is even, it is open for further research for a more general expression. My appreciation goes to Dr. Makanjuola, S. O. for his valuable insight into this area and his immense contribution towards this work.

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